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PRIMARY STRATEGIES OF SIMPLE PURSUIT IN DIFFERENTIAL GAMES ON TWO-SIDED PLANE FIGURES†

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Differential games of simple pursuit in which the game space is a plane two-sided figure are considered. The players may go from one side to the other through an edge of the figure or cuts in it. Examples of such spaces are a circular disk, a rectangle and a two-sided plane with a circular cut. Conditions for the existence of singular pursuit trajectories are investigated. A sufficient condition is found (in terms of the geometrical parameters of the figure and the ratio of the players' speeds) for the existence of a simple pursuit strategy.

So-called simple motion is often used to simulate the motions of manoeuvring objects in differential game theory. Simple motion, in the conventional terminology [1], is that of the non-inertial point moving with controlled velocity, usually subject to symmetrical (spherical) constraints. The solution of many game problems with simple motions turns out to be technically less difficult than in problems with more complex dynamics. For example, the solution of problems of approach and pursuit with simple motion in Euclidean space is comparatively elementary. It is also worth mentioning two differences between more complex differential-game models and simple games in Euclidean space. These models are described by more complex, non-linear equations of motion or, confining ourselves to simple motions, the game space in such games has a more complicated geometry and is a (Riemannian) manifold, possibly with a boundary or singular points, where the smoothness of the manifold breaks down. The properties of non-linearity to some degree absorb the metric tensor of the manifold, but the character of the dynamical equations permits effective use of geometrical methods.

The solution of game problems with complex non-linear dynamics is a fairly difficult mathematical problem. A rigorous formalization of such problems of the differential game theory and the development of effective methods for solving them, are associated in contemporary science with the names of N. N. Krasovskii [2–4], representatives of the school of control theory that he founded [5, 6] and other specialists [7, 8].

In this paper differential games of pursuit with simple motion are considered on two-dimensional manifolds. The payoff in the game is the pursuit time; the capture radius is assumed to be zero.

The optimal pursuit time for players performing simple motion in a Euclidean plane (space) is computed using a simple and well-known formula: it is equal to the ratio of the initial distance between the players to the difference of their speeds [1]. This formula is also true for games on smooth two-dimensional surfaces (manifolds), provided the initial distance between them is sufficiently small. The players' optimal trajectories are determined by the shortest curve (geodesic) connecting the players. Under these conditions both player, pursuer and evader, move along this common geodesic curve. Such a strategy of pursuer (evader) will be called a primary pursuit (evasion) strategy.

However, the existence of configurations (i.e. positions of both points or players), for which two geodesic curves of different lengths exist, may generate singular pursuit trajectories, which are envelopes of a family of geodesics, as well as other types of trajectories. As follows from previous results [9], singular trajectories necessarily arise in a pursuit game on a cone [10, 11]. If, for example, the game space is a

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right circular cylinder, the existence of two geodesic curves does not affect the optimality of a primary strategy, and there are no singular trajectories in such a game.

The purpose of this paper is to investigate the conditions for the occurrence of singular pursuit trajectories, to formulate the sufficient conditions under which, for a given game space and given ratio of the players' speeds, the primary strategies are optimal, and to survey and analyse the solutions of a variety of game problems in different game spaces/manifolds. The main element of the geometrical analysis is the construction of a three-dimensional manifold of players' positions with two equal geodesic curves. Other geometrical methods for solving game problems were effectively investigated in [12, 13].

The sufficient condition for the globality of primary strategy is written as an inequality in terms of tangent unit vectors at the ends of geodesic curves [11]. The ratio of the players' speeds is expressed as a function of the eccentricity of an ellipse, determining the set of ellipses with optimal pursuit along a primary geodesic.

1. GAME SPACE AND DYNAMICS

Consider a differential game with the participation of two controllable points/players – the pursuer P and the evader E. The points perform simple motion, that is, they may simultaneously change the direction of their velocities, which are bounded in magnitude by positive numbers 1 and ν $(0 < \nu < 1)$, respectively.

The game space in which the points are moving is a two-sided plane figure M, possibly unbounded, with a convex boundary. The points may move from one side of the manifold M to the other at points of the boundary. A similar game space is a manifold without a boundary which is, however, not smooth because of the presence of an edge – the boundary of a figure. We shall also consider in this paper examples in which M is a smooth manifold or has a boundary. In all cases the manifold M is a two-dimensional surface or figure, considered in three-dimensional Euclidean space, and the length of a curve on M is understood in the sense of Euclidean length in the enveloping three-dimensional space.

Note that the non-smoothness of the manifold does not give rise to any serious difficulties in the analysis of the problem. The set M may be approximated by a smooth manifold: for example, an ellipse may be considered as a degenerate ellipsoid with vanishing minor axis. The specific feature of the problem is the presence of two or more minimal geodesics connecting the players in some subset of positions.

The equations of motion of the players are given by the following relations and constraint

$$\dot{x} = u, \quad \dot{y} = \upsilon, \quad x, y, u, \upsilon \in R^2 \tag{1.1}$$

$$u_1^2 + u_2^2 \leq 1, \quad \upsilon_1^2 + \upsilon_2^2 \leq \nu^2 \tag{1.2}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are the Euclidean (local) coordinates of the players P and E in plane coinciding with the plane of the figure and having a rectangular Cartesian system of coordinates. In addition to Eqs (1.1) we must also indicate on which side of the figure (say, upper or lower) the points P or E are situated. In addition, conditions preventing the players from moving outside the figure M must be added at points of the boundary of constraint (1.2).

2. PRIMARY SOLUTION

A primary solution is a solution that is identical in structure with the solution of the game in the Euclidean plane. The pursuit time (assuming that the capture radius is zero) under such conditions is equal to the ratio of the initial distance between the players to the difference in their speeds:

$$V(x, y) = L(x, y)/(1 - \nu) \tag{2.1}$$

The initial distance for cases when the players are situated on the same side (the geodesic curve is a segment) or on different sides (the geodesic curve is a broken line consisting of two segments) of the figure M, respectively, is given by the formulae

$$L(x, y) = |x - y|, \quad L(x, y) = |x - d| + |y - d| \tag{2.2}$$

where $d = (d_1, d_2)$ is the corner point of the geodesic, that is, the point at which the curve goes from one side of the figure M to the other.

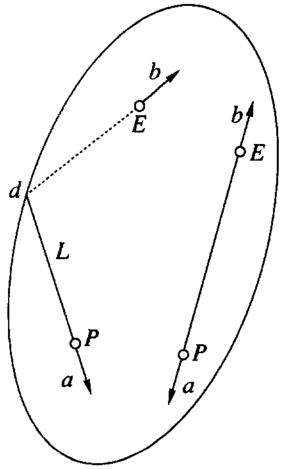

Fig. 1

The optimal behaviour of the players in a primary domain consists of their motion along the shortest geodesic connecting them. The corresponding primary strategies are given by the formulae

$$u(x, y) = -a(x, y)(a(x, y) = \partial L/\partial x), \quad \upsilon(x, y) = \nu b(x, y)(b(x, y) = \partial L/\partial y) \tag{2.3}$$

where a and b are the unit outward tangent vectors to the geodesic curve at the points P and E (directed along the appropriate segment; Fig. 1). A direct check will verify that the modulus of the gradient of the function $L(x, y)$ with respect to each of the variables x and y is 1. In other words, this function satisfies the two eikonal equations. Note that when $L(x, y)$ is differentiated, the derivatives of the vector $d = d(x, y)$ vanish, thanks to the Weierstrass–Erdmann conditions [14] at the corner point of the extremal (the law stating that the "angle of incidence equals the angle of reflection" for extremals). In general form, the expressions for the unit tangent vectors are derived from the formula for the first variation of a geodesic functional [9, 11, 14].

A domain $D_I \subset M \times M$ of the phase space $M \times M$ in which the primary pursuit time (2.1) is optimal is said by definition to be primary. Note that in formula (2.1) we have in mind the shortest geodesic defined by the relations

$$L(x, y) = \min_{\alpha \in A} S(x, y, \alpha) \tag{2.4}$$

where $S(x, y, \alpha)$ is the family of possible local minima of the variational problem of the geodesic length and α is the parameter of the family, which runs through some set A. In the problems considered, the set A usually consists of two elements.

We shall show the time (2.1) is guaranteed for player P in the whole phase space $M \times M$, that is, optimal capture occurs either in time (2.1) or earlier. For fixed α, the total derivative with respect to time of the function $S(x, y, \alpha)$ along trajectories of system (1.1) is

$$\dot{S} = \langle a, u\rangle + \langle b, \upsilon\rangle (a = S_x, b = S_y)$$

The maximum rate of decrease of the distance between the players that player P can achieve is

$$\min_u \dot{L} = \min_u \min_{\alpha \in A} \dot{S} = -1 + \min_{\alpha \in A^*} \langle b, \upsilon\rangle \leq -1 + \nu \tag{2.5}$$

where A^* is the subset of the set A on which the minimum (2.4) is reached (see [15]); the inequality is established using constraint (1.2) imposed on the vector ν.

Inequality (2.5) proves the required time estimate.

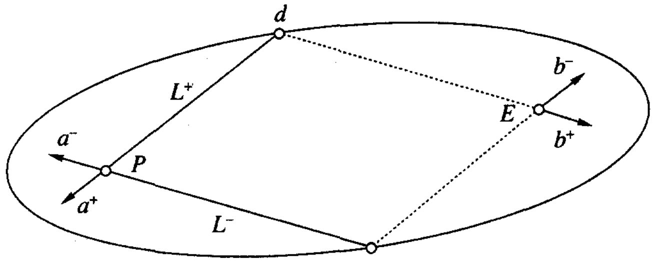

Fig. 2

The analogous statement for the evader – guaranteeing the time (2.1) – is not valid, because of the maximin property, namely, the inequality

$$\max_{v} \dot{L} = \max_{v} \min_{\alpha \in A^*} \dot{S} \leq \min_{\alpha \in A^*} \max_{v} \dot{S}$$

3. OTHER TYPES OF OPTIMAL TRAJECTORIES

We will represent the phase space $M \times M$ as a sum

$$M \times M = D_1 + D_2$$

where, by definition, the optimal pursuit time is equal to (2.1) in D_1 and strictly less than (2.1) in the secondary domain D_2. In known examples of simple pursuit problems, the following types of optimal trajectory are encountered [9, 11, 16–21]. The boundary Γ_{12} between domains D_1 and D_2 consists of singular trajectories, each of the players moving along the envelope of a family of geodesic curves. The domain D_2 may consist of two parts, $D_2 = D_2^0 + D_3$, such that in D_2^0 each of the two players moves along his "own" geodesic curve, which is not the same as the curve connecting them. In D_3 only the pursuer is moving along a geodesic curve, while the evader may move in a non-unique fashion along an arbitrary trajectory. The value of the game in D_3 is the solution of a certain optimal control problem for player P only.

In pursuit problems on a plane with an obstacle or cut, there are also fragments of trajectories on the boundary of the obstacle or on the curve of the cut [16, 17, 19].

Depending on the geometrical properties of the game space, the sets D_3 or D_2 may turn out to be empty. In the latter case the whole game space coincides with the domain D_1 and the primary pursuit strategy is optimal.

4. MANIFOLD OF POSITIONS WITH NON-UNIQUE GEODESIC

If both players are on the same side of the figure M (the game space), a unique geodesic of minimum length connecting them exists – the segment PE. If the players are on different sides of the figure M, then there is a set of positions for which there are two or more shortest geodesic curves of equal length. Positions for which both players are on the boundary of the figure M must be included in that case. Then two geodesic curves – the segments PE exist on different sides of the figure M.

For an ellipse and a rectangle, for example, it is not difficult to verify that the situation is as described. For an ellipse, non-uniqueness mean the existence of exactly two geodesics; only for four positions, when the players are at different foci of the ellipse (for a circle – in the centre), do infinitely many equal geodesics exist. For rectangle, there are positions for which the number of geodesics is two, three, and four (in the case of square).

Denote the manifold of positions with non-unique geodesic – a submanifold of the phase space $M \times M$ – by Γ, $\Gamma \subset M \times M$. Analysis shows that Γ is a three-dimensional manifold with a boundary. Its interior points correspond to positions with two geodesics. In the neighbourhood of these points the length of the geodesic curve has two local minima, which we denote by L^+ and L^- (Fig. 2). For such interior points of Γ we have the equality

$$\Gamma\colon L^+(x, y) = L^-(x, y) \tag{4.1}$$

The boundary points of the manifold Γ may have one, three, or more geodesics.

5. THE MANIFOLD B

A part $\Gamma_1 \subset \Gamma$ of the manifold Γ is, in the conventional terminology [1], a dispersal surface, that is, from each point of Γ_1 one can draw two optimal trajectories of primary pursuit with the same pursuit time (2.1). Primary pursuit takes place along either of the two geodesics; the actual curve is chosen by player P.

Depending on the geometrical properties of the space M, the set Γ_1 may coincide with the entire manifold Γ, in which case the primary solution is a solution of the game in the entire space. If that is not the case, the existence of two geodesics enables player P to manoeuvre in such a way that in some subset $D_2 \subset M \times M$ of the phase space $M \times M$ the optimal pursuit time is strictly less than (2.1).

We shall find the necessary conditions for the primary solution to be optimal in the entire space. These conditions relate the geometrical characteristics of the figure M and the parameter ν. By definition, the set A in formula (2.4) in the neighbourhood of points of the manifold Γ consists of two elements, $A = \{+, -\}$, and the formula itself becomes

$$L(x, y) = \min[L^+(x, y), L^-(x, y)] \tag{5.1}$$

Relation (4.1) means that the minimum (5.1) is reached at both elements, that is, the sets A and A^* are identical. Then the derivative of (5.1) along trajectories of Eqs (1.1) may be written for points of the set Γ as

$$\dot{L} = \min[\langle a^+, u\rangle + \langle b^+, v\rangle, \langle a^-, u\rangle + \langle b^-, v\rangle] \tag{5.2}$$

where $a^\pm$ and $b^\pm$ are outward unit normals to the geodesic curves L^+ and L^- at the points P and E (Fig. 2), analogous to those introduced in (2.3).

As follows from the derivation of formula (2.5), player P can guarantee himself time (2.1), using the strategy $u = -a^+$ or $u = -a^-$ up to capture. We now prescribe for player P in a position in the set Γ the control

$$u = -(a^+ + a^-)/|a^+ + a^-| \tag{5.3}$$

which means that the player is moving not along one of the geodesics but choose a direction between them. Substituting this control into formula (5.2), we obtain the estimate

$$\dot{L} \le -|a^+ + a^-|/2 + \nu|b^+ + b^-|/2 \tag{5.4}$$

where we have used the equality

$$|a^+ + a^-|^2 = \langle a^+ + a^-, a^+ + a^-\rangle = 2(1 + \langle a^+, a^-\rangle)$$

and also the fact that the maximum

$$\max_v \min[\langle b^+, v\rangle, \langle b^-, v\rangle]$$

is reached on the vector

$$v = \nu(b^+ + b^-)/|b^+ + b^-|$$

Comparing relations (2.5) and (5.4), we can conclude that the control (5.3) is preferable for player P, if, at some point of the manifold Γ, we have

$$|a^+ + a^-| - \nu|b^+ + b^-| > 2(1 - \nu) \tag{5.5}$$

Relations (5.2)–(5.5) are "instantaneous" in nature and refer to the initial instant of time. However, they ensure the existence of a sufficiently small time interval in which player P is able to reduce the distance between the players more rapidly than (2.5), thus improving the time (2.1).

The surface Γ_1 lies in the domain D_1 where the time (2.1) is optimal, and therefore condition (5.5) cannot be satisfied there. Thus, at points of the surface Γ_1,

$$\Gamma_1 : L^+ = L^-, \quad |a^+ + a^-| - \nu|b^+ + b^-| \le 2(1-\nu)$$

For the primary solution to be globally optimal, the set (5.5) must be empty. In other words, the boundary B of the manifold Γ_1

$$B: L^+ = L^-, \quad |a^+ + a^-| - \nu|b^+ + b^-| = 2(1-\nu) \tag{5.6}$$

cannot have points in common with the interior of the manifold Γ.

6. EXAMPLES OF THE CONSTRUCTION OF THE MANIFOLDS Γ AND B

We will now present the results of an investigation of the manifold Γ and its boundary $\partial\Gamma$ for a few figures and other game spaces.

A cone and other surfaces of revolution. Let the game space be a surface of revolution defined in a three-dimensional rectangular Cartesian system of coordinates by

$$M = \{(x_1, x_2, x_3) : kx_3^2 = x_1^2 + x_2^2 + m^2, x_3 \ge 0\} \tag{6.1}$$

where k and m are positive constants. When $m = 0$ Eq. (6.1) defines a right circular cone; other values of m give one of the sheets of a hyperboloid of revolution.

The length of a curve on the surface M is defined as its Euclidean length in (x_1, x_2, x_3) space. Let (x_1, x_2, x_3) denote the position of player P, and (y_1, y_2, y_3) that of player E. As local coordinates of the players we shall use the quantities $x = (x_1, x_2)$ and $y = (y_1, y_2)$ – the projections of the points P and E onto the corresponding coordinate plane. The equations of motion in these variables have the form (1.1), but the constraints take a more general form

$$\sqrt{\langle G(x)u, u\rangle} \le 1, \quad \sqrt{\langle G(y)v, v\rangle} \le \nu$$

where $G(x)$ is the metric tensor obtained by projection.

By virtue of the circular symmetry of the space M, the manifold Γ also possesses symmetry. A point of Γ is characterized by the property that the players are situated in a common plane passing through the x_3 axis. As local coordinates on Γ one can take the angle β of rotation of the plane (e.g. about the x_1 axis) and the distances R and r of players P and E, respectively, from the vertex $(0, 0, m/k)$.

Numerical investigation [20] has shown that for fixed β the points of the manifold Γ are defined in the (R, r) plane by a curve $\partial\Gamma$, which asymptotically approaches the coordinate axes (Fig. 3). Points above the curve belong to Γ and points below it correspond to positions for which a single geodesic curve connects the players. As the curve $\partial\Gamma$ is approached from above, the two geodesic curves tend to one another, so that on the curve itself there is single geodesic.

The merging of the two geodesics at points of the curve $\partial\Gamma$ means that for positions in the set $\partial\Gamma$ the points are conjugate to one another in the variational problem of geodesic length on the surface M [14]

$$L(x, y) = \min_{\xi(\sigma)} \int_{\sigma_0}^{\sigma_1} \sqrt{\langle G(\xi)\dot{\xi}, \dot{\xi}\rangle}\, d\sigma, \quad \xi(\sigma_0) = y, \quad \xi(\sigma_1) = x \tag{6.2}$$

This makes it possible to propose an algorithm for finding the points of the curve $\partial\Gamma$ [21]. The extremal passing through the vertex of the surface M must be considered as nominal. In the two-dimensional local coordinate plane this will be a straight line, which may be taken as one of the coordinate axes. One then fixes on that line a discrete sequence of positions of one player, subsequently determining the conjugate focus relative to the functional (6.2) for each such position.

Another possible mechanism for the transition from two geodesics to one is the appearance on $\partial\Gamma$ of three curves of equal length. This happens in the case of a rectangle. It is also possible for a surface

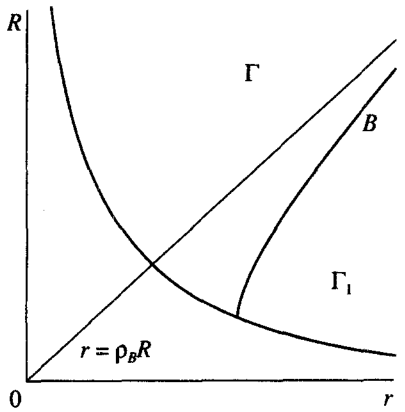

Fig. 3

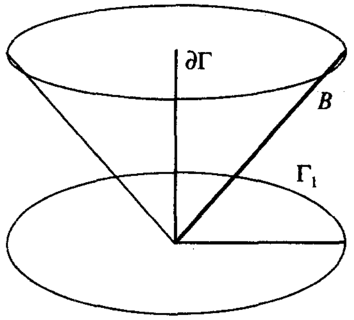

Fig. 4

of revolution generated by a suitable curve. Global (numerical) analysis is necessary to ascertain the mechanism of transition.

As to the numerical construction of the manifold B defined by formula (5.6), it has been shown to have a boundary ∂B on the curve $\partial\Gamma$, and curve B itself has an asymptote

$$r = \rho_B R, \quad \rho_B = \rho_B(\nu, \alpha)$$

which represents the manifold B for a cone (Fig. 3). The dependence of the coefficient ρ_B on the ratio of the speeds ν and the half-angle α of the plane development of the cone have been computed explicitly [9].

As $m \to 0$ the configuration approaches that of a cone. Then the curve $\partial\Gamma$ tends to a right angle whose sides are the positive semi-axes, the curve B tends to its asymptote, and the point ∂B tends to the origin.

The manifolds Γ, $\partial\Gamma$, B, ∂B may be obtained completely (with a suitable third coordinate) by rotating the curves in Fig. 3 about the abscissa or ordinate axis. The boundary $\partial\Gamma$ for a cone then recalls a "button" – a plane with half a straight line attached to it (Fig. 4). The manifold B also turns out to be a cone; it is diffeomorphic to the game space M [9].

A two-sided plane angle. The consideration of this physical game space is equivalent to the case of a cone, which may be deformed in two ways into a plane figure, namely: by cutting it along a generator and "unfolding" it onto a plane; or by folding it along two opposite generators into a plane two-sided angle. The angle α will then equal half the angle of the "unfolded" cone, and moreover $0 < \alpha < \pi$. In both cases, the length of a geodesic curve is preserved in the deformation, and the optimal trajectories are in one-to-one correspondence.

It is convenient to distinguish between "acute" cones, for which the parameters (ν, α) satisfy the condition

$$0 \le \nu \le 1 - \sin\alpha, \quad 0 \le \alpha \le \pi/2 \tag{6.3}$$

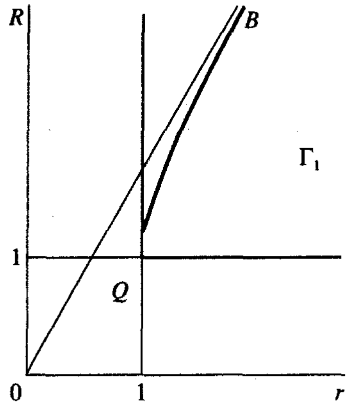

Fig. 5

and all other (obtuse) cones. For acute cones, the sets D_3, D_2^0 are not empty, and if the pursuit begins at points of the domain D_3, the evader, provided he behaves in the optimal fashion, will be captured at the vertex of the cone. For obtuse cones, the set D_3 is empty, but D_2 is not. The manifolds Γ and B have the structure shown in Fig. 4.

A two-sided plane with a disk-shaped cut. In this problem [19], the players may move on both sides of the plane, crossing from one side to the other through the boundary of the removed disk-shaped fragment of the plane. A solution of the problem may be constructed on the basis of a pursuit problem on a (one-sided) plane with a disk-shaped obstacle [16, 17].

Let R and r denote the distances of players P and E, respectively, from the centre O of the disk, whose radius we will assume to be equal to 1; denote the angle between the segments OP and OE by φ. The manifold Γ is defined by the equation $\varphi = \pi$, and its subset Γ_1 by the conditions

$$\varphi = \pi, \quad \sqrt{1-R^{-2}} - \nu\sqrt{1-r^{-2}} \le 1-\nu$$

If the inequality sign on the right is replaced by an "equals" sign, we obtain the equation of the manifold B. The curve B is shown in Fig. 5; it is asymptotic to the straight line $r = \sqrt{\nu}\, R$.

A right angle with vertex at the point Q together with the curve B is the generator whose rotation about the ordinate axis yields the manifolds Γ and B.

As is obvious from the construction, the manifold B is not empty for any of the players' speeds considered.

To obtain the corresponding generator in the original problem, one must take the symmetric generator in Fig. 5 relative to the abscissa axis and glue the point Q to its image.

A two-sided plane punctured at two points. The game space is a two-sided plane punctured at two points (i.e. with two holes of zero diameter) $D_1(0, 1)$, $D_2(0, -1)$, where the players may cross to the other side of the plane. This problem [19] essentially uses the solution of the pursuit game on a plane with a segment-shaped obstacle (the fragment of a wall) [16]. The lengths of the two geodesic curves connecting players on different sides of the plane, and the manifold Γ, are given by the equalities

$$L^{\pm}(x, y) = \frac{1}{1-\nu}\left(\sqrt{x_1^2+(x_2\pm 1)^2} + \sqrt{y_1^2+(y_2\pm 1)^2}\right)$$

$$L^{+}(x, y) = L^{-}(x, y)$$

The manifold B is defined in the set Γ by the additional equality

$$g(x_1, x_2) - \nu g(y_1, y_2) = 1-\nu$$

$$g(\xi, \eta) = \frac{1}{\sqrt{2}}\left(1 + \frac{\xi^2+\eta^2-1}{\sqrt{\xi^2+(\eta+1)^2}\sqrt{\xi^2+(\eta-1)^2}}\right)$$

The manifold B is not empty for any of the player speeds under consideration.

A polygon. It can be shown that if the game space is a convex polygon (e.g. a triangle, rectangle, etc.), the set B is not empty, and thus a primary strategy cannot be globally optimal. Indeed, in a sufficiently small neighbourhood of any vertex of the polygon M the optimal trajectories will be the same as in the problem of a plane angle (cone). In the latter case, there are always singular trajectories. All we can say is that if the angle at the vertex is small enough (in the sense of condition (6.3)), player P can "drive" player E into a corner there, that is, force an encounter at that point.

A plane two-sided strip and a cylinder. The set B is empty, since because of symmetry relative to permutation of the points P and E, the tangent vectors in Eqs (5.5) and (5.6) satisfy the conditions

$$|a^+ + a^-| = |b^+ + b^-| < 2$$

for which Eq. (5.6) is not possible for any ν.

An ellipse. In the case of an ellipse, depending on the values of parameters ν and ε, the set B may be either empty or non-empty; ε is the eccentricity of the ellipse:

$$\varepsilon = c/a, \quad 0 < \varepsilon < 1, \quad c^2 = a^2 - b^2$$

where a and b are the semi-major and semi-minor axes.

In Fig. 6 we show the curve in the (ν, ε) plane separating the domains A_1 and A_2 to which "prolate" and "rounded" ellipses correspond. For ellipses defined by parameters from the set A_1 the set B is empty, and a primary strategy is globally optimal. The same is true for ellipses corresponding to the separating curve, for which, though the set B is non-empty, inequality (5.5) does not hold.

Points (ν, ε) of the set A_1 satisfy the condition

$$A_1: \max_{(P,E)}(|a^+ + a^-| - \nu|b^+ + b^-|) \le 2(1-\nu), \quad (P,E) \in \Gamma$$

The boundary of the set A_1, at whose points this inequality becomes an equality, has been determined numerically.

As to the set Γ for an ellipse, it does not possess rotational symmetry, as in several of the examples considered above. Such symmetry holds for a disk. Before constructing the set Γ for a disk, we describe it for the case of a sphere. It is readily seen that, for any position of player P on a sphere, only one position of the player exists – at the opposite end the diameter – at which the uniqueness of the geodesic of minimum length connecting the players breaks down. Hence it follows that the set Γ is diffeomorphic to a sphere, that is, it will be a sphere in suitable coordinates. The set Γ for two-dimensional game spaces is generally three-dimensional, so that the case of a (two-dimensional) sphere is degenerate in that sense. However, for a disk, which is the limiting case of an ellipsoid with one zero semiaxis, this degeneracy is removed.

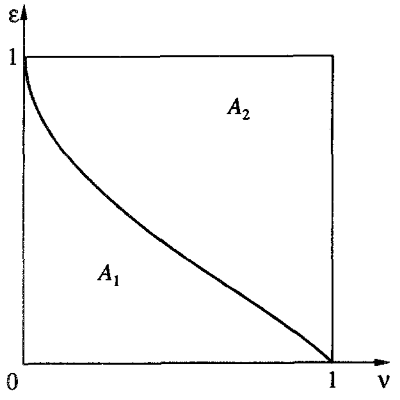

Fig. 6

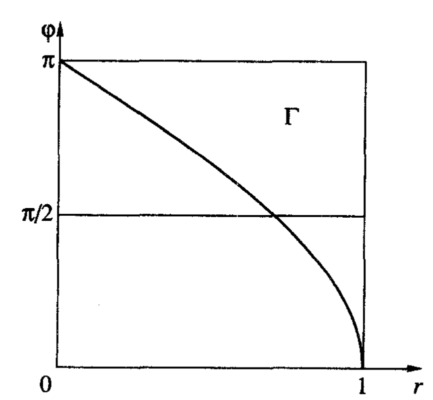

Fig. 7

To describe the set Γ, we note, first of all, that two geodesics of equal length exist on the disk only if the points P and E lie at the same distance r for centre, $0 \le r \le 1$. Denote the angle between the radii on which the points P and E are situated by φ, $0 \le \varphi \le \pi$. In the (r, φ) plane the points of the set Γ are situated in the upper part of the rectangle in Fig. 7. The lower boundary of the domain Γ is the curve $r = \cos(\varphi/2)$. The whole set Γ is obtained by reflecting the figure shown in Fig. 7 in the abscissa axis and rotating the resulting combined figure about the ordinate axis.

7. CONCLUSION

In game problems of simple pursuit, for all the game spaces considered, with the exception of a cylinder and an ellipse, the set B is always non-empty and an optimally operating pursuer cannot be confined to primary pursuit strategies. In the case of a cylinder (a two-sided plane strip) the set B is always empty and player P will achieve the optimal result only by using a primary strategy. In the case of an ellipse both situations may occur, depending on the parameters of the problem. As can be seen from the case of a polygon, the corner points on the boundary of a two-sided plane figure generate singular pursuit trajectories. For a primary strategy to be sufficient, the boundary of the figure must be fairly smooth.

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